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A thermodynamic internal variable model for the partition of plastic work into heat and stored energy in metals

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Abstract

The energy balance equation for elastoplastic solids includes heat source terms that govern the conversion of some of the plastic work into heat. The remainder contributes to the stored energy of cold work due to the creation of crystal defects. This paper is concerned with the fraction β of the rate of plastic work converted into heating. We examine the status of the common assumption that β is a constant with regard to the thermodynamic foundations of thermoplasticity and experiments. A general internal-variable theory is introduced and restricted to abide by the second law of thermodynamics. Experimentally motivated assumptions reduce this theory to a special model of classical thermoplasticity. The only part of the internal energy not determined from the isothermal response is the stored energy of cold work, a function only of the internal variables. We show that this function can be inferred from stress and temperature data from a single adiabatic straining experiment. Experimental data from dynamic Kolsky-bar tests at various strain rates yield a unique stored energy function. Its knowledge is crucial for the determination of the thermomechanical response in non-isothermal processes. Such a prediction agrees well with results from dynamic tests at different rates. In these experiments, β is found to depend strongly on both strain and strain rate for various engineering materials. The model is successful in predicting this dependence. Requiring β to

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be constant is thus an approximation of dubious validity.

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1. Introduction

A complete description of thermomechanical processes for elastoplastic solids at the continuum level involves appropriate statements of the first and second laws of thermodynamics, in addition to momentum balance laws and constitutive relations. The statement of energy balance (the first law of thermodynamics) may be viewed as an equation governing the evolution of the temperature field. As such, it involves heat source terms representing heating due to thermomechanical coupling and inelastic dissipation. One of these terms describes heating through recoverable (elastic) deformations. Another term governs the conversion of some part of the non-recoverable plastic work into heat. The remaining part of the plastic work contributes to the storage of internal energy through the creation and rearrangement of crystal imperfections, especially dislocations, but also point defects, stacking faults and twins. This is known as the stored energy of cold work.

This paper is concerned with the fraction, commonly denoted by β, of the rate of plastic work converted into heating. It has been customary in the mechanics literature to assume that β is a constant, typically chosen between 0.8 and 1 (e.g., Belytschko et al., 1991; Clifton et al., 1984; Kapoor and Nemat-Nasser, 1998; Needleman and Tvergaard, 1995; Simo and Miehe, 1992; Wright and Ockendon, 1992; Zhou et al., 1996). This practice is often justified by citing the early pioneering work of G. I. Taylor and his coworkers (Farren and Taylor, 1925; Taylor and Quinney, 1937), who performed the first experiments investigating this issue. Since then, numerous attempts have been made to measure β, resulting in a wealth of experimental information on the stored energy of cold work. This information is primarily obtained by means of quasistatic experiments and features alarming amounts of scatter in data for nominally identical materials. A detailed review of the literature on cold work is provided by Bever et al. (1973) and various experimental techniques to measure it have been compared by Hodowany et al. (1998).

More recently, Mason et al. (1993) made the first attempt to measure β via fully dynamic experiments performed over a wide range of strains and strain rates. The main contribution of this study was to demonstrate experimentally that β can have a substantial dependence on strain and strain rate. This motivated an extensive experimental investigation using a Kolsky pressure bar to measure the stress–strain response and a high-speed infrared detector to measure the temperature during deformation (Hodowany et al., 1998). They present systematic and conclusive experimental evidence of the effect of strain and strain rate on the partition of plastic work into stored energy and heat. The results show dramatic variations of β with both strain and strain rate for engineering materials such as aluminum alloys or titanium.

This observation suggests that assuming β to be a constant in theoretical and numerical studies of coupled thermomechanical processes may result in inaccuracies in predicted strain, stress and temperature fields, especially in the presence of high-rate adiabatic deformations. In particular, thermoplastic instabilities associated with dynamic shear banding (Zhou et al., 1996), dynamic necking, penetration (Camacho and Ortiz, 1997), and high-speed machining (Marusich and Ortiz, 1995) are accompanied by localized strains and strain rates, and substantial heat generation due to thermoplastic coupling, and are thus expected to be influenced by the form of β.

Other problems of practical interest that are affected by this issue address the temperature rise at the tips of dynamically propagating cracks (Zehnder and Rosakis, 1991), pore collapse during dynamic consolidation (Tong and Ravichandran, 1995), and hot spot initiation during mechanical loading of explosives (Dienes, 1996; Howe et al., 1985).

This investigation examines the theoretical foundations underlying the partition of plastic work into heat and stored energy. We seek a rational thermodynamic basis for a version of the energy balance equation that is often taken for granted in the literature and in which β plays a central role.

Section 2 contains a brief summary of the fundamentals of continuum thermodynamics. In Section 3 we describe an internal-variable formulation of classical, rate-dependent thermoplasticity theory. We enforce the restrictions imposed on this general theory by the second law of thermodynamics. The first law of thermodynamics, or energy balance equation, is thereby brought into a form that allows identification of the thermoelastic and inelastic contributions to heat generation. For simplicity the development is one-dimensional; this suffices for a treatment of the Kolsky pressure bar experiments described by Hodowany et al. (1998).

The theory is specialized further in Sections 4 and 5 by means of experimentally motivated assumptions. These concern the dependence of the specific heat and the stress response function on the internal variables and the elastic strain. The main consequence of these assumptions is that the internal energy decouples into a sum of three functions of one variable. These are identified as the elastic energy, which depends solely on elastic strain, the thermal energy, which is a function of temperature, and the stored energy of cold work, which depends only on the internal variable. A similar decomposition holds for the entropy. Physically speaking, the stored energy of cold work is due to the generation and rearrangement of defects during plastic deformation; it depends on the extent of the latter through the internal variable. The inelastic heating term of the energy equation reduces to the excess of the rate of plastic work over the rate of change of the stored energy of cold work. The ratio β of the inelastic heating over the plastic work rate is thus determined by the specific form of the stored energy of cold work and the evolution laws governing plastic flow. As a result, β depends on the history of the thermomechanical process and is not a constant in general.

Evolution laws for the plastic strain and the internal variable are discussed in Section 6 with emphasis on a special class of rate-dependent thermoplastic constitutive models.

In Section 7, we focus on adiabatic homogeneous processes with a view towards modeling the Kolsky pressure bar experiments described by Hodowany et al. (1998). For such processes we find that the temperature and stress histories are determined by the strain history as solutions of a system of ordinary differential equations. However, this is possible only if the stored energy of cold work is a known function of the internal variable. This function should be recognized as playing a constitutive role; without it the model is incomplete and inapplicable to any non-isothermal process involving plastic flow.

We demonstrate that the stored energy of cold work can be obtained from a single adiabatic experimental record of the stress and temperature history, provided that the isothermal response and thermal softening characteristics of the material are known. Using experimental data from Hodowany et al. (1998), we measure the stored energy of cold work for an essentially rate-insensitive aluminum alloy and highly rate-sensitive titanium. For each material, experiments at different strain rates produce the same stored energy of cold work as a function of the internal variable, in a fashion consistent with the theory.

Once the dependence of the stored energy of cold work on plastic strain is established, the complete adiabatic stress and temperature response of these materials is predicted for tests at various levels of strain rate. The results are in excellent agreement with experimental data found in Hodowany et al. (1998).

A detailed study of the fraction β is undertaken in Section 8. For constant-rate adiabatic processes, our approach yields a semi-analytical expression for β as a function of plastic strain, with a parametric dependence on plastic strain rate. For aluminum we find that β has a strong dependence on strain (but is essentially independent of the strain rate). In the case of titanium we find a somewhat less dramatic dependence on strain, but a substantial dependence on strain rate. Both observations show very good agreement with values of β measured experimentally by Hodowany et al. (1998).

For a general, not necessarily adiabatic, thermomechanical process, β is a history-dependent quantity. Requiring it to be a constant is at best an approximation, and is not supported by experimental evidence. A complete constitutive model for thermoplasticity should include a free energy function accounting for the stored energy of cold work (e.g., Chaboche, 1993; Bodner and Lindenfeld, 1995), instead of ad hoc assumptions about β. An expression for the stored energy of cold work may be obtained in various ways. One possibility is to extract it from dynamic experiments as we show in this paper. Alternatively, one might deduce it from a micromechanical model that accounts for the generation and evolution of crystal defects, and keeps track of the energy stored in their configuration. Examples of that approach are studies by Aravas et al. (1990) and Zehnder (1990).

2. General thermodynamic considerations

We recall some of the fundamentals of continuum thermodynamics: see e.g., Truesdell (1984) and Bowen (1989) for general treatments; Carlson (1972) for thermoelasticity; Lubliner (1990) and Antman (1995) for thermoplasticity theories. A thermomechanical process for a solid body can be described by a *vector displacement field* $\mathbf{u}(\mathbf{x}, t)$ and a scalar *absolute temperature field* $\theta(\mathbf{x}, t)$. Here $\mathbf{x}$ is the position vector of a particle in a fixed reference configuration of the body and t is time; we employ a referential (Lagrangian) description.

The thermomechanical process is to conform to momentum balance and to the first and second laws of thermodynamics. These balance laws involve the following fields: *Piola–Kirchhoff stress tensor field* $\boldsymbol{\sigma}$, *internal energy* per unit volume e, *entropy* per unit volume η, *absolute temperature* $\theta > 0$, *heat flux vector* $\mathbf{q}$ and *heat supply* per unit volume[1] r. The local version of the first law of thermodynamics requires that for every thermomechanical process, the above fields satisfy the *energy balance equation*

$$\boldsymbol{\sigma} \cdot \nabla\dot{\mathbf{u}} + \nabla \cdot \mathbf{q} + r = \dot{e}; \tag{1}$$

the term $\boldsymbol{\sigma} \cdot \nabla\dot{\mathbf{u}}$ is the *stress power*. The following local version of the second law of thermodynamics is known in continuum mechanics as the Clausius–Duhem inequality:

$$\dot{\eta} - \nabla \cdot (\mathbf{q}/\theta) - r \geq 0. \tag{2}$$

After use of (1), the second law (2) assumes the alternative form

$$[\theta\dot{\eta} - \dot{e} + \boldsymbol{\sigma} \cdot \nabla\dot{\mathbf{u}}] + \frac{1}{\theta}\mathbf{q} \cdot \nabla\theta \geq 0. \tag{3}$$

The expression within brackets above is called the internal dissipation per unit volume, defined as

$$\delta = \boldsymbol{\sigma} \cdot \nabla\dot{\mathbf{u}} - \dot{e} + \theta\dot{\eta}. \tag{4}$$

This definition divides the entropy production into the sum of internally dissipative and heat-conductive parts.

The second law can also be stated in terms of the Helmholtz free energy per unit volume $\psi = e - \theta\eta$ as follows:

$$-\dot{\psi} - \eta\dot{\theta} + \boldsymbol{\sigma} \cdot \nabla\dot{\mathbf{u}} + \frac{1}{\theta}\mathbf{q} \cdot \nabla\theta \geq 0. \tag{5}$$

The internal dissipation δ satisfies

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[1] These correspond to specific fields (per unit mass) multiplied by the constant reference mass density.

$$\delta = \boldsymbol{\sigma} \cdot \nabla \dot{\mathbf{u}} - \dot{\psi} - \eta \dot{\theta}. \tag{6}$$

By using (1), (6) and the definition of ψ, the energy balance equation can now be written as

$$\delta + \nabla \cdot \mathbf{q} + r = \theta \dot{\eta}. \tag{7}$$

Constitutive laws relating the above fields are the subject of the next section.

3. Constitutive description in terms of internal variables

In this paper we develop a one-dimensional constitutive model in the context of classical thermoplasticity theory. All fields have a one-dimensional spatial dependence on the reference coordinate x. Thermomechanical processes are described by a scalar axial displacement $u=u(x, t)$ and a temperature $\theta=\theta(x, t)$. The scalars of σ, $\varepsilon=u_x$ and q denote axial stress, strain[2] and heat flux, respectively, while $\nabla\theta$ is determined by θ_x; subscripts indicate partial differentiation with respect to the corresponding variable. We employ an internal-variable formulation. This involves introduction of two additional constitutive variables, the plastic strain ε^{p} and the internal variable or hardening variable[3] ξ, which are to obey suitable evolution laws. The constitutive laws for σ, e, η, q and ψ, as well as the evolution laws for the rates $\dot{\varepsilon}^{\mathrm{p}}$ and $\dot{\xi}$, are assumed to be expressible via functions of the same array of variables, namely (ε, ε^{p}, ξ, θ, θ_x), as suggested by the notion of equipresence (Truesdell and Toupin, 1960). Here however, we immediately confine attention to the special case of classical plasticity, in which dependence on ε and ε^{p} appears only through their difference $\varepsilon-\varepsilon^{\mathrm{p}}$. The elastic strain is accordingly defined as

$$\varepsilon^{\mathrm{e}} = \varepsilon - \varepsilon^{\mathrm{p}} \tag{8}$$

so that the usual additive decomposition $\varepsilon=\varepsilon^{\mathrm{e}}+\varepsilon^{\mathrm{p}}$ holds. Specifically, we assume that for every thermomechanical process of the body

$$\sigma = \hat{\sigma}(\varepsilon^{\mathrm{e}}, \xi, \theta, \theta_x),$$

$$e = \hat{e}(\varepsilon^{\mathrm{e}}, \xi, \theta, \theta_x),$$

$$\eta = \hat{\eta}(\varepsilon^{\mathrm{e}}, \xi, \theta, \theta_x),$$

$$\psi = \hat{\psi}(\varepsilon^{\mathrm{e}}, \xi, \theta, \theta_x),$$

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[2] Alternatively, in one-dimensional shear u is the transverse displacement, σ and ε are shear stress and strain, respectively.

[3] A generalization involving multiple internal variables is straightforward.

$$q = \hat{q}(\varepsilon^{\mathrm{e}}, \xi, \theta, \theta_x),$$

$$\dot{\varepsilon}^{\mathrm{p}} = \hat{P}(\varepsilon^{\mathrm{e}}, \xi, \theta, \theta_x),$$

$$\dot{\xi} = \hat{\Xi}(\varepsilon^{\mathrm{e}}, \xi, \theta, \theta_x), \tag{9}$$

the right-hand-sides being constitutive response functions, assumed to be as smooth as desired. The last two equations represent *evolution laws* for the plastic strain and the internal variable. The formulation so far follows well-established internal-variable theories (Coleman and Gurtin, 1967; Rice, 1970, 1971; Lubliner 1972, 1990; Antman, 1995); it leads to a rate-dependent thermoplasticity theory as will become evident in Section 6.

We restrict the constitutive description by requiring that the second law (5) hold for every thermodynamic process. This standard procedure of constitutive reduction is due to Coleman and Noll (1963); for its implications on plasticity see Lubliner (1972, 1990) and Chaboche (1993). Using the chain rule and (8) we find

$$\dot{\psi} = \hat{\psi}_{\varepsilon^{\mathrm{e}}}\dot{\varepsilon} - \hat{\psi}_{\varepsilon^{\mathrm{e}}}\dot{\varepsilon}^{\mathrm{p}} + \hat{\psi}_{\xi}\dot{\xi} + \hat{\psi}_{\theta}\dot{\theta} + \hat{\psi}_{\theta_x}\dot{\theta}_x, \tag{10}$$

where subscripts indicate partial differentiation (e.g., $\hat{\psi}_{\varepsilon^{\mathrm{e}}} = \partial\hat{\psi}(\varepsilon^{\mathrm{e}}, \xi, \theta, \theta_x)/\partial\varepsilon^{\mathrm{e}}$). Substituting the above into (5) and using the evolution laws in (9) for $\dot{\varepsilon}^{\mathrm{p}}$ and $\dot{\xi}$, we obtain

$$\begin{aligned}&[\hat{\sigma}(\varepsilon^{\mathrm{e}}, \xi, \theta, \theta_x) - \hat{\psi}_{\varepsilon^{\mathrm{e}}}(\varepsilon^{\mathrm{e}}, \xi, \theta, \theta_x)]\dot{\varepsilon} - [\hat{\eta}(\varepsilon^{\mathrm{e}}, \xi, \theta, \theta_x) + \hat{\psi}_{\theta}(\varepsilon^{\mathrm{e}}, \xi, \theta,\\ &\quad \theta_x)]\dot{\theta} - \hat{\psi}_{\theta_x}(\varepsilon^{\mathrm{e}}, \xi, \theta, \theta_x)\dot{\theta}_x + [\hat{\psi}_{\varepsilon^{\mathrm{e}}}(\varepsilon^{\mathrm{e}}, \xi, \theta, \theta_x)\hat{P}(\varepsilon^{\mathrm{e}}, \xi, \theta, \theta_x) - \hat{\psi}_{\xi}(\varepsilon^{\mathrm{e}}, \xi,\\ &\quad \theta, \theta_x)\hat{\Xi}(\varepsilon^{\mathrm{e}}, \xi, \theta, \theta_x)] + \frac{1}{\theta}\hat{q}(\varepsilon^{\mathrm{e}}, \xi, \theta, \theta_x)\theta_x \geq 0.\end{aligned} \tag{11}$$

By requiring that (11) should hold for any thermodynamic process, one concludes that the terms multiplying $\dot{\varepsilon}$, $\dot{\theta}$ and $\dot{\theta}_x$ in the above expression must vanish. This implies that

$$\psi = \hat{\psi}(\varepsilon^{\mathrm{e}}, \xi, \theta),$$

$$\sigma = \hat{\sigma}(\varepsilon^{\mathrm{e}}, \xi, \theta) = \hat{\psi}_{\varepsilon^{\mathrm{e}}}(\varepsilon^{\mathrm{e}}, \xi, \theta),$$

$$\eta = \hat{\eta}(\varepsilon^{\mathrm{e}}, \xi, \theta) = -\hat{\psi}_{\theta}(\varepsilon^{\mathrm{e}}, \xi, \theta). \tag{12}$$

In particular, $\hat{\psi}$, $\hat{\sigma}$ and $\hat{\eta}$ are independent of θ_x, whereas $\hat{q}$, $\hat{P}$ and $\hat{\Xi}$ may depend on the entire array of variables (Chaboche, 1993). The second law now reduces to

$$\hat{\psi}_{\varepsilon^{\mathrm{e}}}(\varepsilon^{\mathrm{e}}, \xi, \theta)\hat{P}(\varepsilon^{\mathrm{e}}, \xi, \theta, \theta_x) - \hat{\psi}_{\xi}(\varepsilon^{\mathrm{e}}, \xi, \theta)\hat{\Xi}(\varepsilon^{\mathrm{e}}, \xi, \theta, \theta_x) + \frac{1}{\theta}\hat{q}(\varepsilon^{\mathrm{e}}, \xi, \theta, \theta_x)\theta_x \geq 0. \tag{13}$$

In view of (12), the internal dissipation[4] in (6) is given by

$$\delta = \sigma\dot{\varepsilon}^{\mathrm{p}} - \hat{\psi}_{\xi}\dot{\xi}. \tag{14}$$

After use of (14) and the last of (12), energy balance (7) takes the form

$$q_x + r + \sigma\dot{\varepsilon}^{\mathrm{p}} - \hat{\psi}_{\xi}\dot{\xi} = \theta\dot{\eta} = -\theta\dot{\overline{\hat{\psi}_{\theta}}}.$$

By utilizing the chain rule, rearranging the above yields:

$$q_x + r + \dot{Q}^{\mathrm{e}} + \dot{Q}^{\mathrm{p}} = c\dot{\theta}, \tag{15}$$

where

$$\dot{Q}^{\mathrm{e}} = \theta\hat{\psi}_{\varepsilon^{\mathrm{e}}\theta}(\varepsilon^{\mathrm{e}}, \xi, \theta)\dot{\varepsilon}^{\mathrm{e}} = \theta\hat{\sigma}_{\theta}(\varepsilon^{\mathrm{e}}, \xi, \theta)\dot{\varepsilon}^{\mathrm{e}},$$

$$\dot{Q}^{\mathrm{p}} = \sigma\dot{\varepsilon}^{\mathrm{p}} - [\hat{\psi}_{\xi}(\varepsilon^{\mathrm{e}}, \xi, \theta) - \theta\hat{\psi}_{\xi\theta}(\varepsilon^{\mathrm{e}}, \xi, \theta)]\dot{\xi},$$

$$c = \hat{c}(\varepsilon^{\mathrm{e}}, \xi, \theta) = -\theta\hat{\psi}_{\theta\theta}(\varepsilon^{\mathrm{e}}, \xi, \theta). \tag{16}$$

Here c is the *specific heat*[5] $\dot{Q}^{\mathrm{e}}$ is the heating due to thermoelastic effects and $\dot{Q}^{\mathrm{p}}$ represents the inelastic contribution to the heating. The first term in the expression for $\dot{Q}^{\mathrm{p}}$ is the *rate of plastic work* $\dot{W}^{\mathrm{p}} = \sigma\dot{\varepsilon}^{\mathrm{p}}$. At this point, we are in a position to define β as *the fraction of the plastic work rate* $\dot{W}^{\mathrm{p}}$ *converted into heating* $\dot{Q}^{\mathrm{p}}$, i.e.,

$$\beta = \dot{Q}^{\mathrm{p}}/\dot{W}^{\mathrm{p}}. \tag{17}$$

Upon inspection of (16) and use of the evolution laws for $\dot{\varepsilon}^{\mathrm{p}}$ and $\dot{\xi}$, it becomes evident that β depends in general on the entire list of variables appearing in (9). As a result, for a specific thermomechanical process, β is a history-dependent quantity. As we show in the rest of the paper, forcing it to be a constant is an assumption of an approximate nature that is not supported by experimental evidence.

4. Restrictions on the specific heat and the stress response function

We introduce specializing assumptions on the dependence of the specific heat $\hat{c}$

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[4] One shows from (13) that $\delta \geq 0$ if the evolution laws are independent of θ_x.

[5] More precisely, the specific heat at constant strain and internal variable, times the mass density.

and the stress response function $\hat{\sigma}$ on the variables ε^{e}, ξ and θ. We then deduce representations of the internal energy, Helmholtz free energy and entropy that follow from these.

A number of investigators have recognized the possibility that cold work may alter the specific heat by changing the modes of atomic vibrations through generation of crystal defects. To investigate this, they measured the specific heat of both annealed and heavily plastically deformed samples at temperature ranges well below the *annealing temperature*, i.e., the lowest temperature at which stored energy is released. Differences in specific heat were never more than 1% and often fell below the sensitivity of the recording technique. For an extensive discussion and references see Bever et al. (1973, Section 1.3.4).

Within our theory, the measure of permanent plastic deformation (or cold work) is the internal variable ξ. Accordingly, *we assume that the specific heat is independent of* ξ and invoke the third of (16) to infer

$$c = \hat{c}(\varepsilon^{\mathrm{e}}, \theta) \Longrightarrow \hat{\psi}_{\theta\theta\xi}(\varepsilon^{\mathrm{e}}, \xi, \theta) = 0. \tag{18}$$

Granted enough smoothness, this implies that $\hat{\psi}_{\xi\theta\theta} = 0$, so that $\hat{\psi}_{\xi\theta\theta}$ is linear in θ:

$$\hat{\psi}_{\xi}(\varepsilon^{\mathrm{e}}, \xi, \theta) = \theta f(\varepsilon^{\mathrm{e}}, \xi) + g(\varepsilon^{\mathrm{e}}, \xi) \tag{19}$$

for some functions f and g of two variables.

In addition, we assume that for purely *elastic processes* (for which $\dot{\varepsilon}^{\mathrm{p}} = \dot{\xi} = 0$), the relation between stress and elastic strain does not depend on the amount of cold work or the number of dislocations accumulated through past deformation. For a *linear* elastic response this implies that the elastic moduli are independent of plastic deformation. This assumption is known to be quite accurate for metals. In the present formulation, recalling the second of (12), we assume that the stress response function $\hat{\sigma}$ is independent of ξ, namely,

$$\hat{\sigma}_{\xi}(\varepsilon^{\mathrm{e}}, \xi, \theta) = \hat{\psi}_{\varepsilon^{\mathrm{e}}\xi}(\varepsilon^{\mathrm{e}}, \xi, \theta) = 0. \tag{20}$$

In conjunction with (19) this gives $\hat{\psi}_{\xi\varepsilon^{\mathrm{e}}} = \theta f_{\varepsilon^{\mathrm{e}}} + g_{\varepsilon^{\mathrm{e}}} = 0$. Since this holds for arbitrary θ, it dictates that $f_{\varepsilon^{\mathrm{e}}} = g_{\varepsilon^{\mathrm{e}}} = 0$; (19) thus reduces to

$$\hat{\psi}_{\xi}(\varepsilon^{\mathrm{e}}, \xi, \theta) = \theta f(\xi) + g(\xi). \tag{21}$$

Let $\bar{\eta}(\xi)$ and $\bar{E}(\xi)$ be such that $\bar{\eta}'(\xi) = -f(\xi)$ and $\bar{E}'(\xi) = g(\xi)$ (primes indicate differentiation for functions of one variable). Integrating (21), we deduce a representation for $\hat{\psi}$ that also restricts $\hat{\eta}$, $\hat{\sigma}$ and $\hat{e} = \hat{\psi} + \theta\hat{\eta}$ in view of (12):

$$\hat{\psi}(\varepsilon^{\mathrm{e}}, \xi, \theta) = \hat{\Psi}(\varepsilon^{\mathrm{e}}, \theta) - \theta\bar{\eta}(\xi) + \bar{E}(\xi),$$

$$\hat{\eta}(\varepsilon^{\mathrm{e}}, \xi, \theta) = -\hat{\Psi}_{\theta}(\varepsilon^{\mathrm{e}}, \theta) + \bar{\eta}(\xi),$$

$$\hat{e}(\varepsilon^{\mathrm{e}}, \xi, \theta) = \hat{E}(\varepsilon^{\mathrm{e}}, \theta) + \bar{E}(\xi),$$

$$\hat{\sigma}(\varepsilon^{\mathrm{e}}, \theta) = \hat{\Psi}_{\varepsilon^{\mathrm{e}}}(\varepsilon^{\mathrm{e}}, \theta). \tag{22}$$

Here we set $\hat{E}(\varepsilon^{\mathrm{e}}, \theta) = \hat{\Psi}(\varepsilon^{\mathrm{e}}, \theta) - \theta\hat{\Psi}_{\theta}(\varepsilon^{\mathrm{e}}, \theta)$.

We conclude that if the specific heat $\hat{c}$ and the stress response function $\hat{\sigma}$ are independent of the internal variable ξ, the dependence of the internal energy $\hat{e}$ and entropy $\hat{\eta}$ on ξ occurs *additively* through functions $\bar{E}(\xi)$ and $\bar{\eta}(\xi)$ and *separately* from their dependence on $(\varepsilon^{\mathrm{e}}, \theta)$. The converse is clearly true as well. We identify $\bar{E}(\xi)$ as the *stored energy of cold work* and $\bar{\eta}(\xi)$ as the *entropy of cold work*. Lubliner (1972) obtains essentially the same representation from similar assumptions. Decompositions similar to (22) are sometimes taken for granted (e.g., Chaboche, 1993; Bodner and Lindenfeld, 1995). In the present setting, (22) follows from assumptions on the behavior of functions directly measurable by experiment, namely, $\hat{c}$ and $\hat{\sigma}$.

Under (22), the energy balance equation (15) takes an especially interesting form. Substituting (22) into (16) we obtain for the inelastic heating

$$\dot{Q}^{\mathrm{p}} = \sigma\dot{\varepsilon}^{\mathrm{p}} - \bar{E}'(\xi)\dot{\xi} = \dot{W}^{\mathrm{p}} - \dot{\bar{E}}. \tag{23}$$

This is considerably simpler than its prior version (16) and has a transparent physical interpretation: *the inelastic heating equals the difference of the plastic work rate* $\sigma\dot{\varepsilon}^{\mathrm{p}} = \dot{W}^{\mathrm{p}}$ and *the rate of change of the internal energy of cold work* $\dot{\bar{E}} = \bar{E}'(\xi)\dot{\xi}$.

Substitution of (22) into (16) for the thermoelastic heating yields

$$\dot{Q}^{\mathrm{e}} = \theta\hat{\Psi}_{\varepsilon^{\mathrm{e}}\theta}(\varepsilon^{\mathrm{e}}, \theta)\dot{\varepsilon}^{\mathrm{e}} = \theta\hat{\sigma}_{\theta}(\varepsilon^{\mathrm{e}}, \theta)\dot{\varepsilon}^{\mathrm{e}} = \theta M(\varepsilon^{\mathrm{e}}, \theta)\dot{\varepsilon}^{\mathrm{e}}, \tag{24}$$

where $M(\varepsilon^{\mathrm{e}}, \theta) = \hat{\sigma}_{\theta}(\varepsilon^{\mathrm{e}}, \theta)$ is the *stress-temperature coefficient*. Eq. (24) has a form associated with purely *thermoelastic* response of the material and does not involve the internal variable ξ. Observe the decoupling of inelastic effects from thermoelastic ones in (23) and (24) in comparison to (16).

The version of energy balance (15) appropriate under the present assumptions is

$$q_x + r + \sigma\dot{\varepsilon}^{\mathrm{p}} - \bar{E}'(\xi)\dot{\xi} + \theta M(\varepsilon^{\mathrm{e}}, \theta)\dot{\varepsilon}^{\mathrm{e}} = \hat{c}(\varepsilon^{\mathrm{e}}, \theta)\dot{\theta}. \tag{25}$$

After use of (22) in (13) we also record the corresponding version of the second law:

$$\sigma\dot{\varepsilon}^{\mathrm{p}} - \bar{E}'(\xi)\dot{\xi} + \theta\bar{\eta}'(\xi)\dot{\xi} + \frac{1}{\theta}q\theta_x \geq 0. \tag{26}$$

5. Further restrictions on the constitutive law

At this point we follow common practice and specialize the model further: *we assume that the specific heat is independent of the elastic strain* ε^{e}, so that

$$c = \hat{c}(\theta). \tag{27}$$

This is almost invariably assumed in the literature. Accordingly, in view of (22) and (16), $\hat{\Psi}_{\varepsilon^{\mathrm{e}}\theta\theta} = 0$, so that after integration we conclude that $\hat{\Psi}(\varepsilon^{\mathrm{e}}, \theta) = \Psi(\theta) - \theta\tilde{\eta}(\varepsilon^{\mathrm{e}}) + \tilde{E}(\varepsilon^{\mathrm{e}})$ in terms of suitable functions of one variable. As a result, (22) reduces to

$$\begin{aligned}
&\hat{\psi}(\varepsilon^{\mathrm{e}}, \xi, \theta) = \Psi(\theta) - \theta[\tilde{\eta}(\varepsilon^{\mathrm{e}}) + \bar{\eta}(\xi)] + \tilde{E}(\varepsilon^{\mathrm{e}}) + \bar{E}(\xi),\\
&\hat{\eta}(\varepsilon^{\mathrm{e}}, \xi, \theta) = -\Psi'(\theta) + \tilde{\eta}(\varepsilon^{\mathrm{e}}) + \bar{\eta}(\xi),\\
&\hat{e}(\varepsilon^{\mathrm{e}}, \xi, \theta) = E(\theta) + \tilde{E}(\varepsilon^{\mathrm{e}}) + \bar{E}(\xi),\\
&\hat{\sigma}(\varepsilon^{\mathrm{e}}, \theta) = \tilde{E}'(\varepsilon^{\mathrm{e}}) + \theta M(\varepsilon^{\mathrm{e}}), \quad M(\varepsilon^{\mathrm{e}}) = -\tilde{\eta}'(\varepsilon^{\mathrm{e}}).
\end{aligned} \tag{28}$$

Here $E(\theta) = \Psi(\theta) - \theta\Psi'(\theta)$. The entropy and internal energy each *decompose additively* into three single-variable functions of θ, ε^{e} and ξ, respectively. In particular, $E(\theta)$ is the *thermal energy*, $\tilde{E}(\varepsilon^{\mathrm{e}})$ and $\tilde{\eta}(\varepsilon^{\mathrm{e}})$ are the *elastic energy and entropy*, $\bar{E}(\xi)$ and $\bar{\eta}(\xi)$ are the *stored energy and entropy of cold work*, respectively. Under assumption (27), the stress response function in the fourth of (28) is linear in θ. The stress-temperature coefficient $M(\varepsilon^{\mathrm{e}}) = -\tilde{\eta}'(\varepsilon^{\mathrm{e}})$ depends only on elastic strain.

Further restrictions can now be introduced. Assuming a *linear thermoelastic stress response*, one specializes (28) so that $\tilde{E}''(\varepsilon^{\mathrm{e}}) = \mu = \text{const.}$ and $-\tilde{\eta}'(\varepsilon^{\mathrm{e}}) = M = \text{const.}$; here $\mu > 0$ and M are the *elastic modulus* and the *stress-temperature coefficient*, respectively. The *thermal expansion coefficient* is $\alpha = -M/\mu$. If the reference configuration is chosen to be stress-free at a reference temperature θ_0, the last of (28) specializes to

$$\hat{\sigma}(\varepsilon^{\mathrm{e}}, \theta) = \mu[\varepsilon^{\mathrm{e}} - \alpha(\theta - \theta_0)]. \tag{29}$$

In addition, one often assumes a *constant specific heat* $\hat{c}(\theta) = c = \text{const.}$; this determines $\Psi(\theta) = -c\theta\log(\theta/\theta_0)$ up to an inessential linear function[6] of θ. In view of the assumptions leading to (29), one easily infers that (28) reduces to

$$\begin{aligned}
&\hat{\psi}(\varepsilon^{\mathrm{e}}, \xi, \theta) = -c\theta\ \log(\theta/\theta_0) + \mu[\varepsilon^{\mathrm{e}2}/2 - (\theta - \theta_0)\alpha\varepsilon^{\mathrm{e}}] + [\bar{E}(\xi) - \theta\bar{\eta}(\xi)],\\
&\hat{\eta}(\varepsilon^{\mathrm{e}}, \xi, \theta) = c[\log(\theta/\theta_0) + 1] + \mu\alpha\varepsilon^{\mathrm{e}} + \bar{\eta}(\xi),\\
&\hat{e}(\varepsilon^{\mathrm{e}}, \xi, \theta) = c\theta + \mu[\varepsilon^{\mathrm{e}2}/2 + \theta_0\alpha\varepsilon^{\mathrm{e}}] + \bar{E}(\xi),
\end{aligned}$$

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[6] Which does not affect any of the field questions.

$$\hat{\sigma}(\varepsilon^{\mathrm{e}}, \theta) = \mu[\varepsilon^{\mathrm{e}} - \alpha(\theta - \theta_0)]. \tag{30}$$

Energy balance (25) takes the form

$$q_x + r + \sigma\dot{\varepsilon}^{\mathrm{p}} - \bar{E}'(\xi)\dot{\xi} - \mu\alpha\theta\dot{\varepsilon}^{\mathrm{e}} = c\dot{\theta}. \tag{31}$$

For completeness we also specify the heat flux q. We adopt the simplest possible Fourier heat-conduction law: $\hat{q}(\varepsilon^{\mathrm{e}}, \xi, \theta, \theta_x) = k\theta_x$, where $k > 0$ is a constant *heat conductivity coefficient*. The energy equation (31) then reduces to an inhomogeneous heat equation:

$$c\dot{\theta} - k\theta_{xx} = \sigma\dot{\varepsilon}^{\mathrm{p}} - \bar{E}'(\xi)\dot{\xi} - \mu\alpha\theta\dot{\varepsilon}^{\mathrm{e}} + r. \tag{32}$$

In this equation, $\dot{Q}^{\mathrm{p}} = \sigma\dot{\varepsilon}^{\mathrm{p}} - \bar{E}'(\xi)\dot{\xi}$ is the inelastic heating, $\dot{W}^{\mathrm{p}} = \sigma\dot{\varepsilon}^{\mathrm{p}}$ is the plastic work rate and $\dot{Q}^{\mathrm{e}} = -\mu\alpha\theta\dot{\varepsilon}^{\mathrm{e}}$ is the thermoelastic heating. In terms of these quantities and β defined in (17), the energy equation becomes

$$c\dot{\theta} - k\theta_{xx} = \beta\dot{W}^{\mathrm{p}} + \dot{Q}^{\mathrm{e}} + r. \tag{33}$$

This frequently employed form does not explicitly involve the stored energy of cold work $\bar{E}$, about which little information is available. Instead, it is customary to assume that β is a material constant[7] with $0.8 \leq \beta \leq 1$. This assumption furnishes a way for the temperature field to be calculated, but is subject to criticism on both theoretical and experimental grounds, as discussed in Sections 7 and 8.

In the above equations, the rates $\dot{\xi}$ and $\dot{\varepsilon}^{\mathrm{p}}$ abide by evolution laws [the last two relations in (9)] that are made explicit in the next section. The only functions that remain to be specified are the stored energy $\bar{E}$ and entropy $\bar{\eta}$ of cold work. The latter does not enter the energy equation. The determination of the stored energy of cold work $\bar{E}$ from experiments is taken up in Section 7.

6. Connection with rate-dependent thermoplasticity

In this section we specify evolution laws that connect the theory developed above to standard semi-empirical thermoplastic models; see Meyers (1994, Chap. 13).

Suppose that the stress response function $\sigma = \hat{\sigma}(\varepsilon^{\mathrm{e}}, \theta)$ is invertible for fixed θ, so that $\varepsilon^{\mathrm{e}} = \hat{\sigma}^{-1}(\sigma, \theta)$; this is the case for (29). Assume also that the evolution laws for ε^{p} and ξ in (9) do not depend on θ_x. They can be written in the form

$$\dot{\varepsilon}^{\mathrm{p}} = P(\sigma, \xi, \theta), \quad \dot{\xi} = \Xi(\sigma, \xi, \theta). \tag{34}$$

We now specialize (34) further by assuming the following properties:

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[7] For example, Belytschko et al. (1991), Kapoor and Nemat-Nasser (1998), Clifton et al. (1984), Simo and Miehe (1992), Wright and Ockendon (1992), and Zhou et al. (1996).

(i) the functions $P(\sigma, \xi, \theta)$, $\Xi(\sigma, \xi, \theta)$ are continuous and piecewise-smooth;
(ii) there exists a smooth, positive *yield-stress function* $\tau(\xi, \theta) > 0$, such that;

$$\Xi(\sigma, \xi, \theta)\begin{cases} = 0 & \text{for } \mid \sigma \mid \leq \tau(\xi, \theta), \\ > 0 & \text{for } \mid \sigma \mid > \tau(\xi, \theta). \end{cases} \tag{35}$$

(iii) the internal variable ξ is identified as *accumulated plastic strain*, namely;

$$\dot{\xi} = \mid \dot{\varepsilon}^{\mathrm{p}} \mid \Leftrightarrow P(\sigma, \xi, \theta) = \mathrm{sign}(\sigma)\Xi(\sigma, \xi, \theta). \tag{36}$$

(iv) the yield stress exhibits *thermal softening*, i.e., decreases with increasing temperature;

$$\frac{\partial}{\partial \theta}\tau(\xi, \theta) < 0.$$

(v) the yield stress does not decrease with increasing permanent deformation at a fixed temperature (no strain softening).

$$\frac{\partial}{\partial \xi}\tau(\xi, \theta) \geq 0.$$

Assumptions (ii) and (iii) imply that $\sigma\dot{\varepsilon}^{\mathrm{p}} > 0$ whenever $\dot{\varepsilon}^{\mathrm{p}} \neq 0$. In the sequel we restrict attention to plastic flow under monotonic loading with $\sigma > \tau > 0$, so that $\dot{\varepsilon}^{\mathrm{p}} = \dot{\xi} > 0$. Then $P(\sigma, \xi, \theta) = \xi(\sigma, \xi, \theta)$ in view of (iii). Under suitable initial conditions we then have for all times

$$\xi = \varepsilon^{\mathrm{p}}. \tag{37}$$

Consider next the following specific example for $\Xi(\sigma, \xi, \theta)$. Let

$$\langle x \rangle = \begin{cases} x & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

Let $H(x)$ be a smooth, monotone-increasing function such that:

$$H' > 0, \quad H(0) = 0. \tag{38}$$

After recalling (35) we specify $\Xi(\sigma, \xi, \theta)$ as follows:

$$\Xi(\sigma, \xi, \theta) = H\left(\left\langle \frac{\mid \sigma \mid}{\tau(\xi, \theta)} - 1 \right\rangle\right). \tag{39}$$

This is known as Perzyna's overstress model (Perzyna, 1966). For monotonic loading, in view of (37), inversion of (39) furnishes

$$\sigma = \tau(\varepsilon^{\mathrm{p}}, \theta)[1 + H^{-1}(\dot{\varepsilon}^{\mathrm{p}})]. \tag{40}$$

Thus, during plastic flow the stress coincides with a specific function of plastic strain, temperature and plastic strain rate, called the *flow stress*. Suppose

$$\tau(\varepsilon^{\mathrm{p}}, \theta) = \sigma_0(\varepsilon^{\mathrm{p}}) f(\theta - \theta_0), \quad h(\dot{\varepsilon}^{\mathrm{p}}) = [1 + H^{-1}(\dot{\varepsilon}^{\mathrm{p}})], \tag{41}$$

where θ_0 is the reference temperature, σ_0 is a non-decreasing function and f is a decreasing function with $f(0) = 1$. Then the flow stress in (40) becomes:

$$\sigma = \sigma_0(\varepsilon^{\mathrm{p}}) f(\theta - \theta_0) h(\dot{\varepsilon}^{\mathrm{p}}). \tag{42}$$

This special type of dependence of the flow stress on plastic strain, temperature and plastic strain rate is commonly assumed on the basis of empirical arguments (e.g., Johnson et al., 1983; Klopp et al., 1985; Zerilli and Armstrong, 1987).

The factor $\sigma_0(\varepsilon^{\mathrm{p}})$ embodies strain hardening. It is measured from a quasistatic test at constant temperature θ_0. Requirement (iv) ensures that flow stress decreases with increasing temperature as observed in experiments. The thermal softening factor $f(\theta - \theta_0)$ can be obtained from a series of isothermal tests at different constant temperatures and very low strain rates, $\dot{\varepsilon}^{\mathrm{p}} \leq 10^{-3}\ \mathrm{s}^{-1}$. The increasing dependence of flow stress on strain rate $\dot{\varepsilon}^{\mathrm{p}}$ is a consequence of (38) and the last of (41). The strain-rate hardening factor $h(\dot{\varepsilon}^{\mathrm{p}})$ is obtained from the dependence of the initial yield stress (at $\varepsilon^{\mathrm{p}} = 0$) on $\dot{\varepsilon}^{\mathrm{p}}$. This is measured from a series of dynamic experiments at different strain rates. The product

$$S(\varepsilon^{\mathrm{p}}, \dot{\varepsilon}^{\mathrm{p}}) = \sigma_0(\varepsilon^{\mathrm{p}}) h(\dot{\varepsilon}^{\mathrm{p}}) \tag{43}$$

is the dynamic isothermal flow stress. For a description of experimentally motivated models of the form (42), see Meyers (1994, Chap. 13).

7. Adiabatic thermomechanical processes

We start by considering the conditions prevailing in the Kolsky pressure bar experiments described by Hodowany et al. (1998). Consistent with the classical treatment of these uniaxial compression experiments, deformations are idealized as homogeneous, so that $\varepsilon_x = 0$. Due to the high strain rates and short duration of the experiments, heat loss through conduction, convection, or radiation is neglected in comparison to thermoplastic heating; accordingly we set $q = r = 0$ in the energy equation (31). This also implies that $\theta_x = 0$ and $\theta_{xx} = 0$ in (31) and (32). We thus assume the process to be adiabatic and homogeneous. The strain history $\varepsilon = \varepsilon(t)$ of the specimen is viewed as given; the strain rate $\dot{\varepsilon} > 0$ is approximated as constant.

Under these circumstances, energy balance (31), the evolution law (39) together with (37) and (29) yield a non-linear system of differential equations for the histories $\theta(t)$ and $\xi(t) = \varepsilon^{\mathrm{p}}(t)$ for a given strain history $\varepsilon(t)$:

$$\begin{aligned} &\dot{\xi} = \Xi(\sigma, \xi, \theta), \\ &\dot{\theta} = (1/c)[\sigma - \bar{E}'(\xi)]\Xi(\sigma, \xi, \theta) - (\mu\alpha/c)\theta[\dot{\varepsilon} - \Xi(\sigma, \xi, \theta)] \quad \text{where} \\ &\quad \sigma = \mu[(\varepsilon - \xi) - \alpha(\theta - \theta_0)]. \end{aligned} \tag{44}$$

In view of assumptions (i)–(iv) on Ξ — in particular (39) — when the third equation is substituted into the first two, the right-hand-sides are continuous and piecewise-smooth functions of ξ, θ and t; explicit dependence on t is through the given function $\varepsilon(t)$. The standard theory of systems of ordinary differential equations then guarantees a unique solution for $\theta(t)$ and $\xi(t)$, hence, for $\sigma(t)$, subject to the initial conditions

$$\theta(0) = \theta_0, \quad \xi(0) = 0.$$

In addition to the material parameters μ, c, α, and the specific choice of $\Xi(\sigma, \xi, \theta)$ described in (39)–(42), the stored energy of cold work $\bar{E}(\xi)$ must be specified in order to determine the stress and temperature history for an adiabatic homogeneous process. Observe that the entropy of cold work $\bar{\eta}(\xi)$ appearing in (30) does not enter (44).

We now consider an approximate version of system (44). We neglect the thermoelastic heating $\dot{Q}^{\mathrm{e}} = -\mu\alpha\theta\dot{\varepsilon}^{\mathrm{e}}$ compared to the thermoplastic heating[8] $\dot{Q}^{\mathrm{p}}$. One way to effect this is to set $\alpha = 0$[9]. Alternatively, one may neglect the elastic strain rate $\dot{\varepsilon}^{\mathrm{e}} = \dot{\varepsilon} - \dot{\varepsilon}^{\mathrm{p}}$ in comparison to $\dot{\varepsilon}^{\mathrm{p}}$. This assumption is in good agreement with experiments; see Hodowany et al. (1998) for details. As a result, in what follows, we set $\dot{\varepsilon}^{\mathrm{p}} = \dot{\varepsilon}$, choose $t = 0$ to correspond to initial yield, and regard the plastic strain history

$$\varepsilon^{\mathrm{p}}(t) = \xi(t) \tag{45}$$

as given. We also define the *adiabatic temperature rise*

$$\vartheta(t) = \theta(t) - \theta_0. \tag{46}$$

In view of (45), this reduces the energy balance [the second of (44)] to the form

$$c\dot{\vartheta} = \sigma\dot{\varepsilon}^{\mathrm{p}} - \bar{E}'(\varepsilon^{\mathrm{p}})\dot{\varepsilon}^{\mathrm{p}} = \dot{Q}^{\mathrm{p}}, \tag{47}$$

from which thermoelastic effects are absent due to the assumption just made. A substitution of the flow-stress expression (42) and (43) into (47) for σ furnishes a first-order differential equation for the temperature-rise history:

$$\dot{\vartheta} = (1/c)[S(\varepsilon^{\mathrm{p}}, \dot{\varepsilon}^{\mathrm{p}})f(\vartheta) - \bar{E}'(\varepsilon^{\mathrm{p}})]\dot{\varepsilon}^{\mathrm{p}}. \tag{48}$$

For a constant-rate process, the solution of this equation expresses the adiabatic temperature rise as a function $\bar{\vartheta}(\varepsilon^{\mathrm{p}}, \dot{\varepsilon}^{\mathrm{p}})$ of ε^{p} with a parametric dependence on $\dot{\varepsilon}^{\mathrm{p}}$. After this is obtained, the stress is determined by the strain and temperature rise from (42) and (43) as

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[8] This approximation is not valid for up to short times after the onset of yielding, but improves very rapidly as the plastic strain rate soon dominates over the elastic one.

[9] This is appropriate for shear in isotropic materials with linear thermoelastic response.

$$\sigma = \bar{\sigma}(\varepsilon^{\mathrm{p}}, \dot{\varepsilon}^{\mathrm{p}}) = S(\varepsilon^{\mathrm{p}}, \dot{\varepsilon}^{\mathrm{p}})f(\bar{\vartheta}(\varepsilon^{\mathrm{p}}, \dot{\varepsilon}^{\mathrm{p}})). \tag{49}$$

7.1. Experimental determination of the stored energy of cold work

We now demonstrate that the stored energy of cold work $\bar{E}$ can be measured from the adiabatic experiments described by Hodowany et al. (1998). This is the only part of the internal energy e that remains unknown in (30) and (32). As observed above, specification of $\bar{E}$ is required in order to determine the response to any thermomechanical process.

Suppose that during a Kolsky-bar test [viewed as an adiabatic homogeneous process at a prescribed constant strain rate $\dot{\varepsilon}^{\mathrm{p}}$; see (45)] the stress and temperature rise are measured as functions $\bar{\sigma}(\varepsilon^{\mathrm{p}}, \dot{\varepsilon}^{\mathrm{p}})$ and $\bar{\vartheta}(\varepsilon^{\mathrm{p}}, \dot{\varepsilon}^{\mathrm{p}})$. Then integration of (47) immediately yields

$$\bar{E}(\varepsilon^{\mathrm{p}}) = \int_0^{\varepsilon^{\mathrm{p}}} \bar{\sigma}(\varepsilon, \dot{\varepsilon}^{\mathrm{p}})\mathrm{d}\varepsilon - c\bar{\vartheta}(\varepsilon^{\mathrm{p}}, \dot{\varepsilon}^{\mathrm{p}}). \tag{50}$$

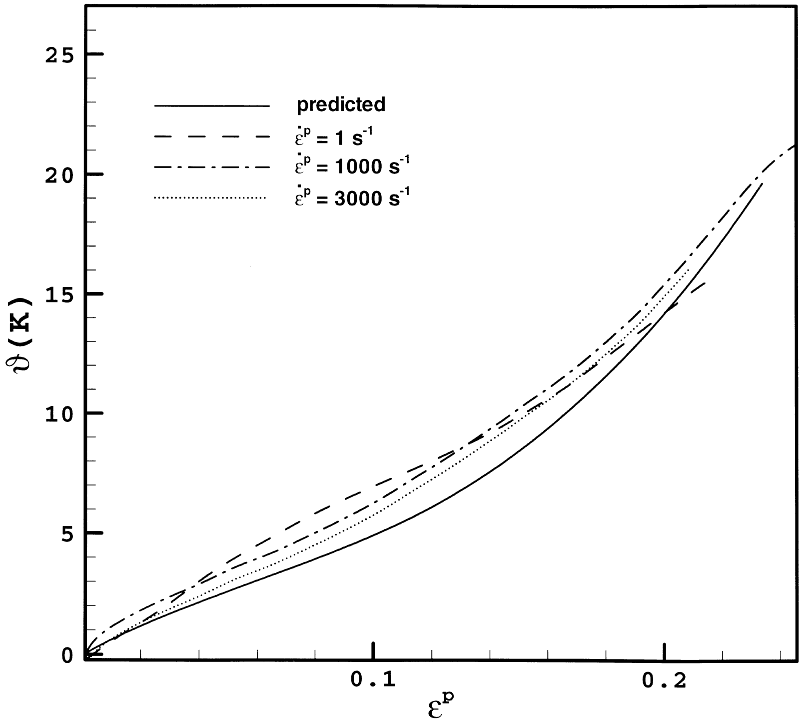

Fig. 1. Adiabatic temperature rise $\vartheta = \bar{\vartheta}(\varepsilon^{\mathrm{p}})$ vs plastic strain ε^{p} for rate-insensitive 2024-T351 aluminum. Dashed and/or dotted lines: experimental measurements at three different strain rates. Solid line: theoretical prediction from Eq. (55).

Here we have chosen $\bar{E}(0) = 0$ without loss of generality. This equation relates the adiabatic stress $\bar{\sigma}(\varepsilon^p, \dot{\varepsilon}^p)$ and the adiabatic temperature rise $\bar{\vartheta}(\varepsilon^p, \dot{\varepsilon}^p)$ to the internal energy of cold work $\bar{E}(\varepsilon^p)$. The first term on the right-hand-side of (50) is the plastic work, that is, the area under the stress-versus-plastic-strain curve extracted from the Kolsky-bar data. The temperature in the second term is directly measured by high-speed infrared detectors during each experiment; see Hodowany et al. (1998) for details.

Both $\bar{\sigma}$ and $\bar{\vartheta}$ are rate-dependent. However, their combination on the right-hand-side of (50) should yield the same value of $\bar{E}$ at a given strain when evaluated using data from experiments at different rates, since the stored energy of cold work cannot depend on plastic strain rate according to the present model.

We turn to the experimental verification of this conclusion. We consider first a 2024-T351 aluminum alloy. This material is essentially *strain-rate-insensitive*; see Hodowany et al. (1998) for details. Fig. 1 displays temperature rise versus plastic strain measured from adiabatic experiments conducted at three different strain rates. The temperature rise was essentially the same in all three tests. This is due

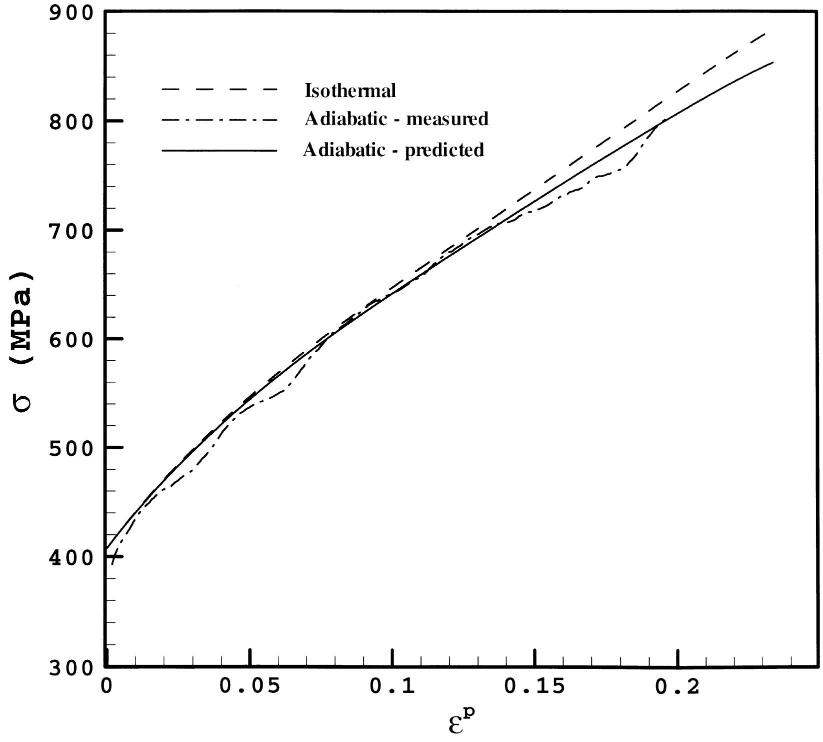

Fig. 2. Stress σ vs plastic strain ε^p for rate-insensitive 2024-T351 aluminum. Dashed line: experimental measurement of the isothermal response $\sigma = \sigma_0(\varepsilon^p)$ at the reference temperature $\theta_0 = 293$ K and $\dot{\varepsilon}^p = 10^{-3}\,\mathrm{s}^{-1}$. Dash–dotted line: experimental adiabatic response $\sigma = \bar{\sigma}(\varepsilon^p)$ at $\dot{\varepsilon}^p = 3 \times 10^3\,\mathrm{s}^{-1}$. Solid line: theoretical prediction of the adiabatic response from Eq. (55).

to the rate independence of the material, as concluded by a prediction based on our constitutive model in Section 7.2 below; see (55). Measurements of stress versus plastic strain are shown in Fig. 2. The isothermal response (dashed line) was measured at the reference temperature $\theta_0 = 293$ K at a low strain rate $\dot{\varepsilon}^{\mathrm{p}} = 10^{-3}\ \mathrm{s}^{-1}$. The adiabatic stress response in high-rate experiments was essentially independent of the strain rate. Results are shown from a single test at $\dot{\varepsilon}^{\mathrm{p}} = 3 \times 10^3\ \mathrm{s}^{-1}$ (dash–dotted line). Evaluation of $\bar{E}(\varepsilon^{\mathrm{p}})$ from (50) using the data of Figs. 1 and 2 is shown in Fig. 3. The various experiments, each of which spans different but overlapping ranges of plastic strain, produce a single curve for $\bar{E}(\varepsilon^{\mathrm{p}})$ for aluminum 2024-T351.

Similar experiments were performed for a highly *rate-sensitive* α-titanium. In this case, the measured adiabatic temperature rise and stress versus plastic strain were found to be strongly rate-sensitive and are shown as dotted curves in Figs. 4 and 5, respectively. However, when stress and temperature data from two different experiments at strain rates, $\dot{\varepsilon}^{\mathrm{p}} = 1\mathrm{s}^{-1}$ and $3 \times 10^3\ \mathrm{s}^{-1}$, were substituted into (50), a single curve resulted for $\bar{E}(\varepsilon^{\mathrm{p}})$, displayed in Fig. 6.

The prediction of a single curve for the stored energy of cold work using experimental data from tests at different strain rates, provides confidence in the

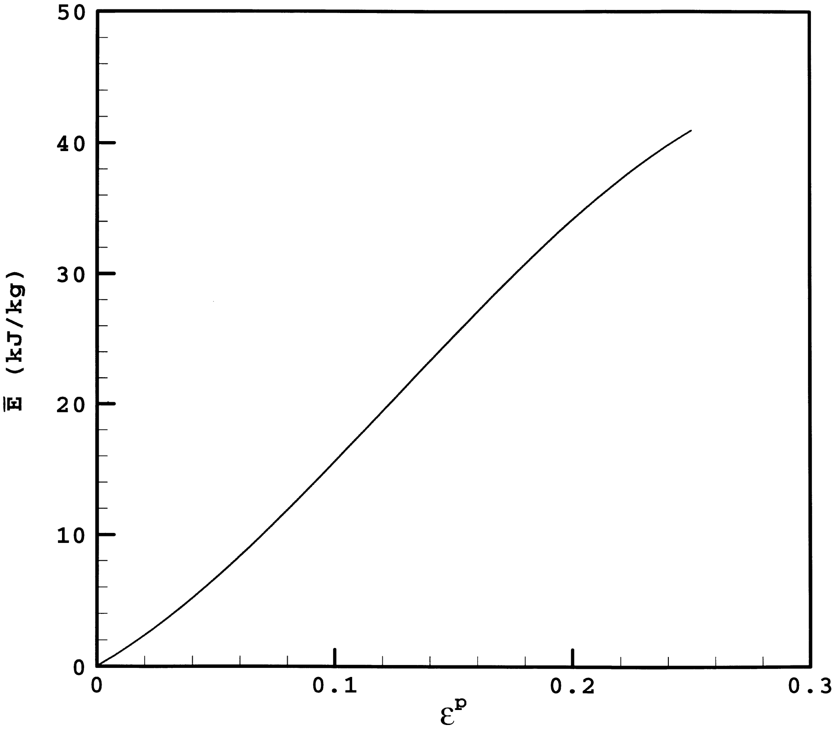

Fig. 3. Stored energy of cold work $\bar{E}(\varepsilon^{\mathrm{p}})$ vs plastic strain ε^{p} for aluminum 2024-T351; calculated from Eq. (50) using the experimental data for $\vartheta = \bar{\vartheta}(\varepsilon^{\mathrm{p}})$ and $\sigma = \bar{\sigma}(\varepsilon^{\mathrm{p}})$ from Figs. 1 and 2.

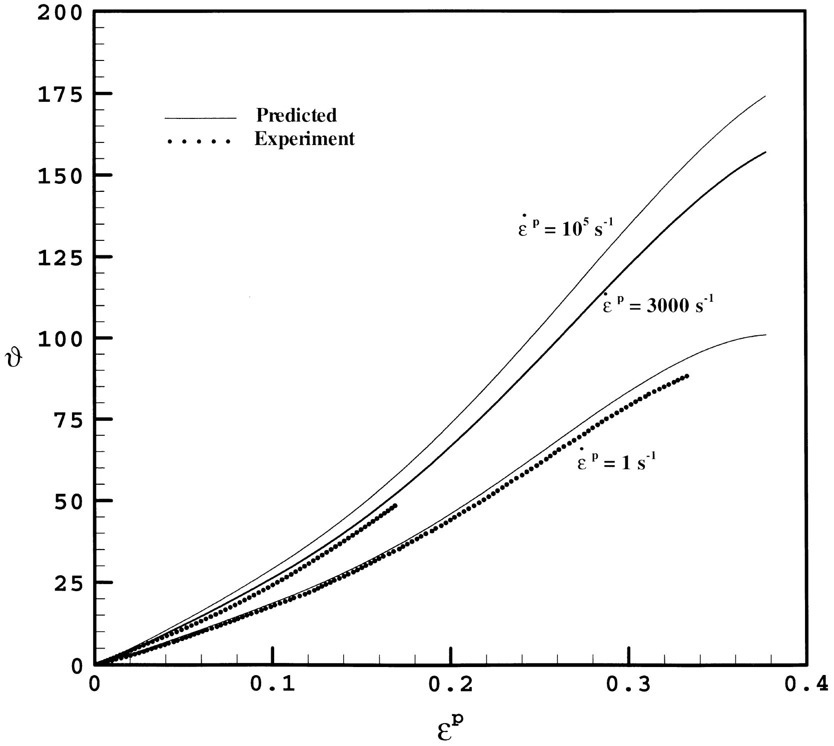

Fig. 4. Adiabatic temperature rise $\vartheta = \bar{\vartheta}(\varepsilon^p, \dot{\varepsilon}^p)$ vs plastic strain ε^p for rate-sensitive α-titanium. Dotted lines: experimental measurements at $\dot{\varepsilon}^p = 1 s^{-1}$ and $3 \times 10^3\ s^{-1}$. Solid line: theoretical prediction from Eq. (53) for $\dot{\varepsilon}^p = 1 s^{-1}$ and $3 \times 10^3\ s^{-1}$, $10^5\ s^{-1}$.

present model. The adiabatic measurements of stress and temperature rise found in Hodowany et al. (1998), together with (50), furnish a straightforward method for the evaluation of $\bar{E}(\varepsilon^p)$. The rate independence of the stored energy of cold work implies that a single adiabatic experiment of the above type suffices for its measurement [10].

7.2. Prediction of the adiabatic thermomechanical response

After the stored energy of cold work $\bar{E}(\varepsilon^p)$ is determined from experiments, the model at hand can predict the stress and temperature rise for any adiabatic straining processes using isothermal constitutive information. To that effect, suppose that the isothermal dynamic flow stress $S(\varepsilon^p, \dot{\varepsilon}^p)$ in (43) is measured from

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[10] This is the case for monotonic uniaxial loading, but probably not for more complex processes, where a stored energy depending on multiple internal variables should be considered. Bodner and Lindenfeld (1995) show that a multi-variable model for the stored energy accounts for its variation in quasistatic cyclic experiments.

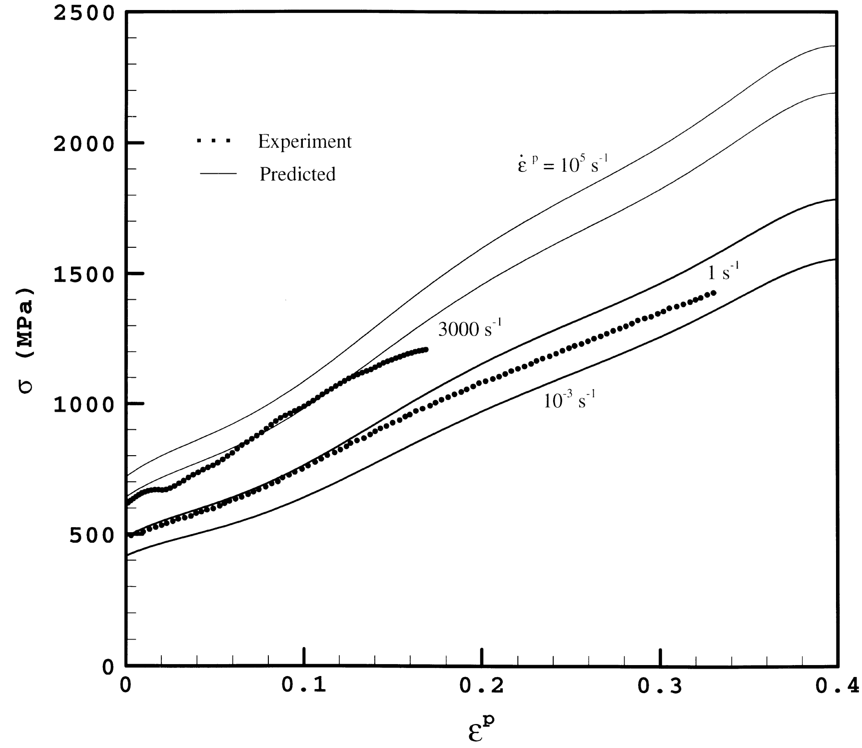

Fig. 5. Adiabatic stress response $\sigma = \bar{\sigma}(\varepsilon^p, \dot{\varepsilon}^p)$ vs plastic strain ε^p for rate-sensitive α-titanium. Dotted lines: experimental measurements at $\dot{\varepsilon}^p = 1 s^{-1}$ and $3 \times 10^3\ s^{-1}$. Solid line: theoretical prediction from Eq. (54) for $\dot{\varepsilon}^p = 1 s^{-1}$ and $3 \times 10^3\ s^{-1}$, $10^5\ s^{-1}$.

dynamic tests at the reference temperature θ_0, as described prior to (43). We assume linear thermal softening in (42) and choose

$$f(\vartheta) = 1 - b\vartheta, \tag{51}$$

where b is the *thermal softening coefficient* (Meyers, 1994) and $\vartheta = \theta - \theta_0$.

With (51) in force, the differential equation (48) for ϑ becomes linear; for a *constant-rate process* ($\dot{\varepsilon}^p = \text{const.}$), it reduces to

$$\frac{\partial \bar{\vartheta}}{\partial \varepsilon^p}(\varepsilon^p, \dot{\varepsilon}^p) + mS(\varepsilon^p, \dot{\varepsilon}^p)\bar{\vartheta}(\varepsilon^p, \dot{\varepsilon}^p) = \frac{1}{c}[S(\varepsilon^p, \dot{\varepsilon}^p) - \bar{E}'(\varepsilon^p)], \qquad m = b/c. \tag{52}$$

The solution, subject to the initial condition $\bar{\vartheta}(0, \dot{\varepsilon}^p) = 0$, is given by

$$\bar{\vartheta}(\varepsilon^p, \dot{\varepsilon}^p) = \frac{1}{c}\int_0^{\varepsilon^p} [S(\varepsilon, \dot{\varepsilon}^p) - \bar{E}'(\varepsilon)] e^{-m\int_{\varepsilon}^{\varepsilon^p} S(\gamma, \dot{\varepsilon}^p) d\gamma} \, d\varepsilon. \tag{53}$$

From this, one immediately obtains a formula for the adiabatic stress $\sigma = \bar{\sigma}(\varepsilon^p, \dot{\varepsilon}^p)$

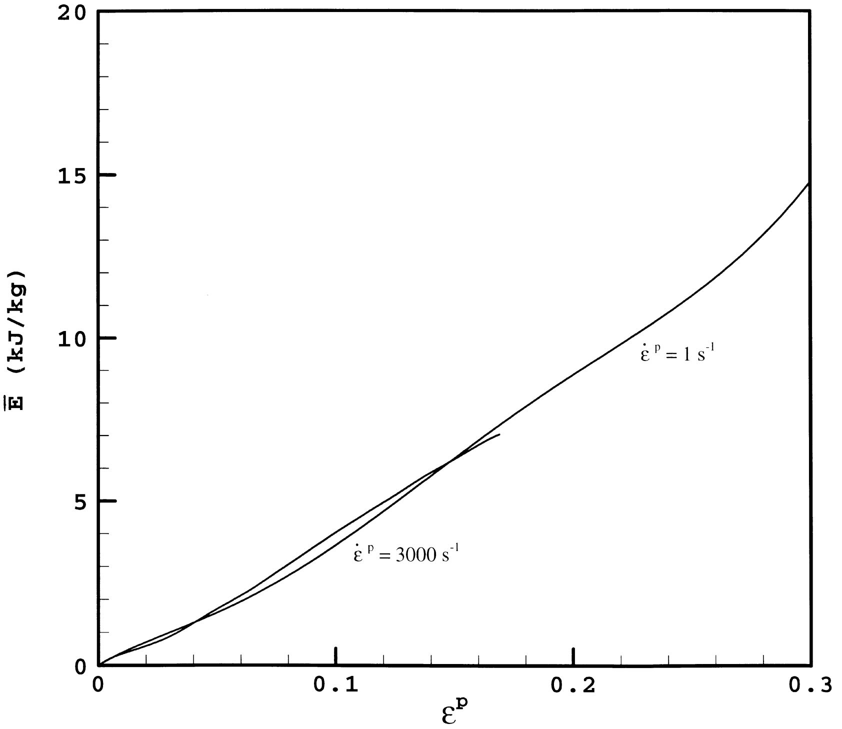

Fig. 6. Stored energy of cold work $\bar{E}(\varepsilon^p)$ vs plastic strain ε^p for rate-sensitive α-titanium; calculated from Eq. (50) using the experimental data for $\vartheta = \bar\vartheta(\varepsilon^p, \dot\varepsilon^p)$ and $\sigma = \bar\sigma(\varepsilon^p, \dot\varepsilon^p)$ at $\dot\varepsilon^p = 1 s^{-1}$ and 3×10^3 s^{-1} from Figs. 4 and 5.

by substitution of (53) into (49), using (51):

$$\bar\sigma(\varepsilon^p, \dot\varepsilon^p) = S(\varepsilon^p, \dot\varepsilon^p) e^{-m\int_0^{\varepsilon^p} S(\varepsilon,\, \dot\varepsilon^p) d\varepsilon} \left[1 + m \int_0^{\varepsilon^p} \bar{E}'(\varepsilon) e^{m\int_0^{\varepsilon} S(\gamma,\, \dot\varepsilon^p) d\gamma}\, d\varepsilon \right]. \qquad (54)$$

This relates the adiabatic stress $\bar\sigma(\varepsilon^p, \dot\varepsilon^p)$ to its isothermal analogue $S(\varepsilon^p, \dot\varepsilon^p)$; these two differ because of thermal softening due to the adiabatic temperature rise $\bar\vartheta$. The latter is due to conversion of plastic work into heat, hence, it involves the stored energy of cold work $\bar{E}(\varepsilon^p)$.

For a *rate-independent material* with $h(\dot\varepsilon^p) = 1$ and $S(\varepsilon^p, \dot\varepsilon^p) = \sigma_0(\varepsilon^p)$ in (42) and (43), $\bar\theta$ and $\bar\sigma$ reduce to rate-independent functions of plastic strain:

$$\bar\vartheta(\varepsilon^p) = \frac{1}{c}\int_0^{\varepsilon^p} [\sigma_0(\varepsilon) - \bar{E}'(\varepsilon)] e^{-m\int_\varepsilon^{\varepsilon^p} \sigma_0(\gamma) d\gamma}\, d\varepsilon, \quad \bar\sigma(\varepsilon^p) = \sigma_0(\varepsilon^p)[1 - b\bar\vartheta(\varepsilon^p)]. \qquad (55)$$

These are compared with experimental measurements for aluminum in Figs. 1 and 2.

We compare the predictions of (53) and (54) with experimental measurements of the adiabatic response at different strain rates for α-Ti. A single experiment at the reference temperature $\theta_0 = 293$ K and at a fixed rate $\varepsilon^{\mathrm{p}} = 1\ \mathrm{s}^{-1}$, was used to obtain $\bar{E}(\varepsilon^{\mathrm{p}})$ (see above, Section 7.1, and Fig. 6). The function $S(\varepsilon^{\mathrm{p}},\ \dot{\varepsilon}^{\mathrm{p}}) = \sigma_0(\varepsilon^{\mathrm{p}})h(\dot{\varepsilon}^{\mathrm{p}})$ is determined as follows: first $\sigma_0(\varepsilon^{\mathrm{p}})$ is measured experimentally for a very low rate $\dot{\varepsilon}_0^{\mathrm{p}} = 10^{-3}\ \mathrm{s}^{-1}$ at θ_0. Then $h(\dot{\varepsilon}^{\mathrm{p}})$ is assumed to be of the form

$$h(\dot{\varepsilon}^{\mathrm{p}}) = 1 + C\ \log_{10}(\dot{\varepsilon}^{\mathrm{p}}/\dot{\varepsilon}_0^{\mathrm{p}}).$$

The constant C is obtained by measuring the variation of the initial yield stress $\sigma_0(0)h(\dot{\varepsilon}^{\mathrm{p}})$ with strain rate through dynamic experiments and fitting to the above form.

Eqs. (53) and (54) were used to predict the adiabatic stress and temperature rise at strain rates of $\dot{\varepsilon}^{\mathrm{p}} = 1\ \mathrm{s}^{-1}$, $3 \times 10^3\ \mathrm{s}^{-1}$ and $10^5\ \mathrm{s}^{-1}$. The results are compared with experimental measurements at the first two of these rates in Fig. 4 for the temperature, and Fig. 5 for the stress.

8. The fraction of plastic work rate converted into heating

Within the specialized constitutive framework developed in Sections 4–7, we re-examine the form of the energy balance equation, paying particular attention to the fraction β of plastic work rate $\dot{W}^{\mathrm{p}} = \sigma\dot{\varepsilon}^{\mathrm{p}}$ converted into inelastic heating $\dot{Q}^{\mathrm{p}}$. In a homogeneous adiabatic process, after neglecting thermoelastic heating, the energy balance reduces to the simple form (47): $c\dot{\theta} = \dot{Q}^{\mathrm{p}} = \beta\dot{W}^{\mathrm{p}}$. Here $\beta = \dot{Q}^{\mathrm{p}}/\dot{W}^{\mathrm{p}}$ defined in (17) may also be expressed as

$$\beta = \frac{c\dot{\theta}}{\sigma\dot{\varepsilon}^{\mathrm{p}}}. \tag{56}$$

This allows β to be extracted from experimental measurements of the temperature and stress versus plastic strain during a Kolsky-bar test, without prior knowledge of the stored energy of cold work.

At the same time, the constitutive model at our disposal allows a theoretical prediction of β. Recalling (23), we have $\dot{Q}^{\mathrm{p}} = \dot{W}^{\mathrm{p}} - \dot{\bar{E}}$. Then (17) and (36) furnish

$$\beta = \frac{\dot{Q}^{\mathrm{p}}}{\dot{W}^{\mathrm{p}}} = 1 - \frac{\bar{E}'(\xi)\dot{\xi}}{\sigma\dot{\varepsilon}^{\mathrm{p}}} = 1 - \frac{\bar{E}'(\xi)}{\mid \sigma \mid} = \hat{\beta}(\sigma,\ \xi). \tag{57}$$

As a result, β is determined by the constitutive law once the stored energy of cold work $\bar{E}(\xi)$ is specified. In (57) one may use (42) for the flow stress and set $\sigma = \sigma_0(\varepsilon^{\mathrm{p}})f(\vartheta)h(\dot{\varepsilon}^{\mathrm{p}})$. This shows that for general thermoplastic processes $\beta = \tilde{\beta}(\varepsilon^{\mathrm{p}},\ \theta,\ \dot{\varepsilon}^{\mathrm{p}})$ depends on plastic strain, strain rate and temperature. Its form is determined by the stored energy of cold work, but also the strain hardening, strain-rate hardening and thermal softening characteristics of the material. For an adiabatic homogeneous process at a constant rate, we may employ (54) for the adiabatic stress $\bar{\sigma}(\varepsilon^{\mathrm{p}},\ \dot{\varepsilon}^{\mathrm{p}})$ in order to obtain an analytical expression for the dependence of β

on strain and strain rate:

$$\beta = \bar{\beta}(\varepsilon^{\mathrm{p}}, \dot{\varepsilon}^{\mathrm{p}}) = 1 - \frac{\bar{E}'(\varepsilon^{\mathrm{p}})}{\bar{\sigma}(\varepsilon^{\mathrm{p}}, \dot{\varepsilon}^{\mathrm{p}})}. \tag{58}$$

For rate-independent materials one uses (55) in place of (54), so that $\beta = \bar{\beta}(\varepsilon^{\mathrm{p}})$. In (58) $\bar{E}(\varepsilon^{\mathrm{p}})$ is obtained from an experiment at a single strain rate, as described in Section 7.1. In Fig. 7, predicted values of β from (58) for aluminum are compared with experimentally obtained values from (56) for strain rates $\dot{\varepsilon}^{\mathrm{p}} = 10^3\ \mathrm{s}^{-1}$ and $3 \times 10^3\ \mathrm{s}^{-1}$. As expected, experimental values of β are essentially rate-insensitive. The model captures well the strong dependence of β on strain; it is found to initially decrease, reach a minimum of less than 0.3 at $\varepsilon^{\mathrm{p}} \approx 0.07$, then increase with strain.

For α-titanium, an analogous comparison is shown in Fig. 8. The dependence of

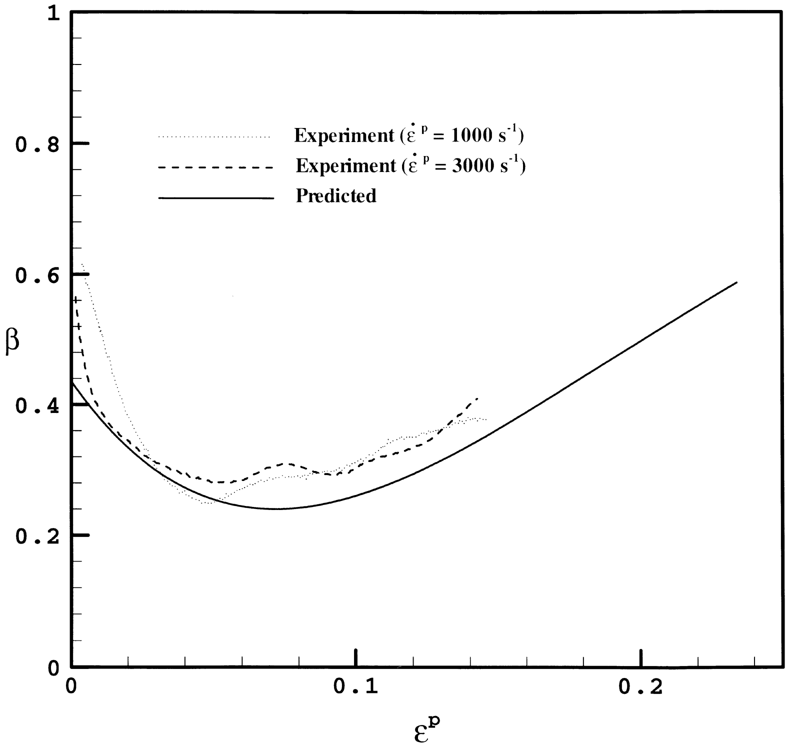

Fig. 7. Fraction β of plastic work rate converted into heating vs plastic strain ε^{p} for rate-insensitive 2024-T351 aluminum. Dashed and dotted lines: experimentally obtained values from Eq. (56) at strain rates $\dot{\varepsilon}^{\mathrm{p}} = 10^3\,\mathrm{s}^{-1}$ and $3 \times 10^3\ \mathrm{s}^{-1}$; see also Hodowany et al. (1998). Solid line: theoretical prediction $\beta = \bar{\beta}(\varepsilon^{\mathrm{p}})$ from Eq. (58), using $\bar{E}(\varepsilon^{\mathrm{p}})$ from Fig. 3 and $\sigma = \bar{\sigma}(\varepsilon^{\mathrm{p}})$ from Eq. (55) (solid line in Fig. 2).

β on strain is somewhat less dramatic than for aluminum. However, the rate dependence of the material is prominent in both the measured and the predicted results. For $\dot{\varepsilon}^{\mathrm{p}} = 1\ \mathrm{s}^{-1}$, β oscillates around an average value of 0.65, while for $\dot{\varepsilon}^{\mathrm{p}} = 3 \times 10^3\ \mathrm{s}^{-1}$ its average is roughly 0.8. This increase is largely due to strain-rate hardening.

These results indicate that aluminum stores a higher fraction of plastic work than α-titanium. In numerical simulations of dynamic penetration of aluminum plates, Camacho and Ortiz (1997) achieved better agreement with experimental results when β was chosen to be zero than with the choice $\beta = 0.9$.

8.1. Concluding remarks

One of the points hopefully clarified by the present study is that β is a history-dependent quantity. The analytical expression obtained when (54) is combined

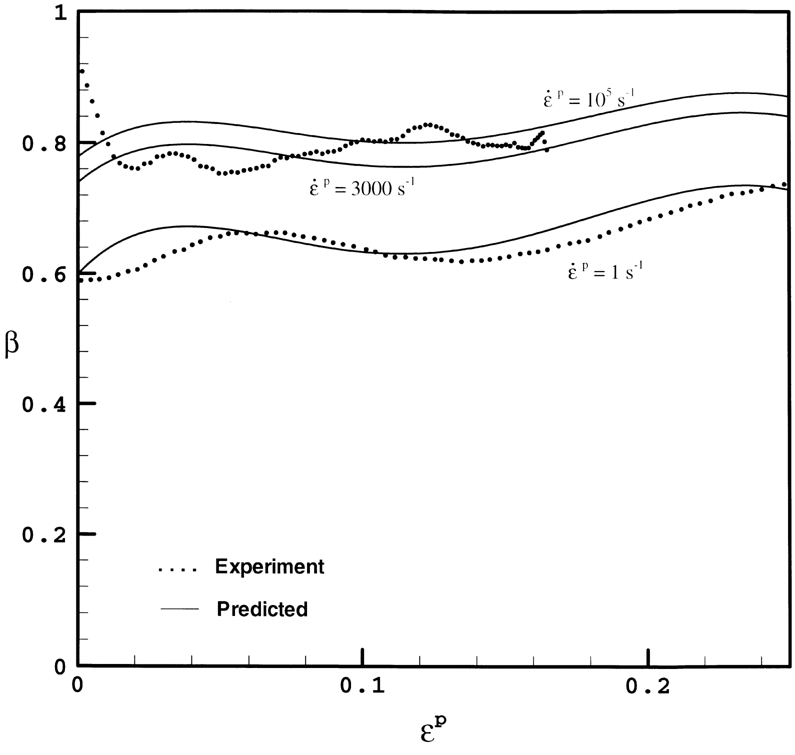

Fig. 8. Fraction β of plastic work rate converted into heating vs plastic strain ε^{p} for rate-sensitive α-titanium. Dotted lines: experimentally obtained values from Eq. (56) at strain rates $\dot{\varepsilon}^{\mathrm{p}} = 1\,\mathrm{s}^{-1}$ and $3 \times 10^3\ \mathrm{s}^{-1}$; see also Hodowany et al. (1998). Solid line: theoretical prediction $\beta = \bar{\beta}(\varepsilon^{\mathrm{p}}, \dot{\varepsilon}^{\mathrm{p}})$ from Eq. (58), using $\bar{E}(\varepsilon^{\mathrm{p}})$ from Fig. 6 and $\sigma = \bar{\sigma}(\varepsilon^{\mathrm{p}}, \dot{\varepsilon}^{\mathrm{p}})$ from Eq. (54) (solid line in Fig. 5).

with (58) is valid only for adiabatic processes at a constant strain rate. For more general processes, it is impossible to express β as a function only of strain and strain rate: the temperature, which affects the stress [hence, β in (57)] through thermal softening, enters in a complex fashion through the solution of the energy equation (32); this is a partial differential equation in the presence of heat conduction.

Within the confines of the present constitutive framework, the assumption that β is constant is inconsistent with the rate independence of the stored energy of cold work, which is a fundamental consequence of thermodynamics. It would seem that the only justification for *a priori* assumptions on β is a lack of information on the stored energy of cold work.

Instead, a consistent constitutive model for thermoplasticity must include an appropriate expression for the stored energy of cold work; we have demonstrated a method for the experimental determination of this function. Its specification, together with the usual evolution laws and the energy balance equation, allows the complete prediction of the material response to arbitrary thermomechanical processes.

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